

Joint Analyticity of the Transformed Field and Dirichlet–Neumann Operator in Periodic Media

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Goals

- 1 Develop a numerical algorithm to record scattered energy in a two-layer periodic structure.
- 2 Prove a theorem on the existence and uniqueness of solutions to a system of partial differential equations which model the interaction of linear waves in periodic layered media.

Overview

- 1 Introduction
- 2 Governing Equations
- 3 High-Order Perturbation of Surfaces
- 4 Wave Scattering
- 5 Joint Analyticity of Solutions
- 6 Conclusion

Maxwell's Equations

As a starting point we consider the time-harmonic Maxwell's equations of electromagnetism in a homogeneous region:

$$\nabla \times \mathbf{E} = i\omega\mu_0\mathbf{H},$$

$$\nabla \times \mathbf{H} = -i\omega\epsilon_0\epsilon\mathbf{E},$$

$$\nabla \cdot \mathbf{E} = 0,$$

$$\nabla \cdot \mathbf{H} = 0.$$

- \mathbf{E} is the electric field, \mathbf{H} is the magnetic field.
- ϵ_0 and μ_0 represent the permittivity and permeability in vacuum.
- ϵ is the complex permittivity, ω is the frequency.

Two-Dimensional Simplifications

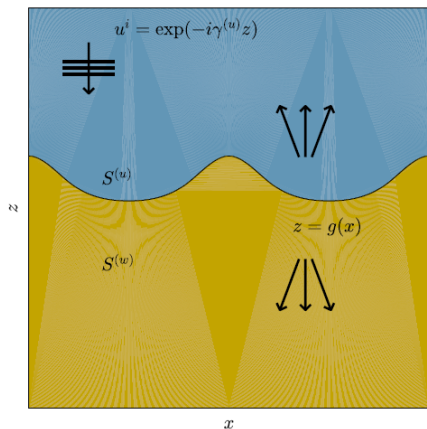
- We choose an interface shaped by $z = g(x, y)$ where the normal is defined by $\mathbf{N} := (-\partial_x g, -\partial_y g, 1)^T$.
- To obtain two-dimensional solutions, we assume the grating shape is invariant in the y -direction:

$$z = g(x),$$

which implies that the interfacial normal becomes

$$\mathbf{N} = \begin{pmatrix} -\partial_x g \\ 0 \\ 1 \end{pmatrix}.$$

The Geometry



A two-layer structure with a periodic interface, $z = g(x)$, separating two material layers, $S^{(u)}$ and $S^{(w)}$.

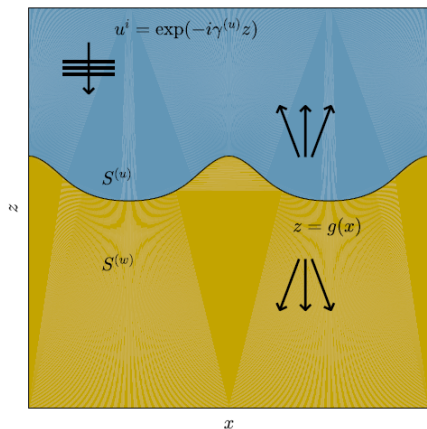
- We consider a y -invariant, doubly layered structure. The interface $z = g(x)$ is d -periodic so that $g(x + d) = g(x)$.
- A dielectric (with refractive index n^u) occupies the domain above the interface

$$S^{(u)} := \{z > g(x)\}.$$

- A material of refractive index n^w is in the lower layer

$$S^{(w)} := \{z < g(x)\}.$$

Incident Radiation



A two-layer structure with a periodic interface, $z = g(x)$, illuminated by plane-wave incidence.

- The structure is illuminated from above by **monochromatic** plane-wave incident radiation of frequency ω .
- We consider the reduced incident fields

$$\mathbf{E}^i(x, z) = e^{i\omega t} \underline{\mathbf{E}}^i(x, z, t),$$

$$\mathbf{H}^i(x, z) = e^{i\omega t} \underline{\mathbf{H}}^i(x, z, t),$$

where the time dependence $\exp(-i\omega t)$ is removed.

- The scattered radiation is “outgoing,” upward propagating in $S^{(u)}$ and downward propagating in $S^{(w)}$.

Governing Equations for Layered Media

- In this 2D setting the time-harmonic Maxwell equations decouple into two scalar Helmholtz problems: Transverse electric (TE) and transverse magnetic (TM) polarizations.
- We define the invariant (y) directions of the scattered (electric or magnetic) fields by $\{\tilde{u}, \tilde{w}\}$ in $S^{(u)}$ and $S^{(w)}$ and seek outgoing/bounded, periodic solutions of

$$\begin{aligned} \Delta \tilde{u} + (k^u)^2 &= 0, & z > g(x), \\ \Delta \tilde{w} + (k^w)^2 &= 0, & z < g(x), \\ \tilde{u} - \tilde{w} &= -\tilde{u}^i, & z = g(x), \\ \partial_N \tilde{u} - \tau^2 \partial_N \tilde{w} &= -\partial_N \tilde{u}^i, & z = g(x). \end{aligned}$$

- $g(x)$ is the grating interface, \tilde{u}^i is the incident radiation.
- $\tau^2 = 1$ in TE, $\tau^2 = (k^u/k^w)^2$ in TM.
- For $q \in \{u, w\}$, $k^q = \omega/c^q$ is the wavenumber.

Governing Equations Without Phase

- We further factor out the phase $\exp(i\alpha x)$ from the fields \tilde{u} and \tilde{w}

$$u(x, z) = e^{-i\alpha x} \tilde{u}(x, z), \quad w(x, z) = e^{-i\alpha x} \tilde{w}(x, z).$$

- With these, our governing equations consist of outgoing/bounded, periodic solutions of

$$\Delta u + 2i\alpha \partial_x u + (\gamma^u)^2 u = 0, \quad z > g(x),$$

$$\Delta w + 2i\alpha \partial_x w + (\gamma^w)^2 w = 0, \quad z < g(x),$$

$$u - w = \zeta, \quad z = g(x),$$

$$\partial_N u - i\alpha(\partial_x g)u - \tau^2 [\partial_N w - i\alpha(\partial_x g)w] = \psi, \quad z = g(x).$$

- $\alpha = k^u \sin(\theta)$, and for $q \in \{u, w\}$, $\gamma^q = k^q \cos(\theta)$.

Artificial Boundaries

- To truncate the bi-infinite problem domain to one of finite size we choose values a and b such that

$$a > |g|_{\infty}, \quad -b < -|g|_{\infty},$$

and define the artificial boundaries $\{z = a\}$ and $\{z = -b\}$.

- In $\{z > a\}$ the Rayleigh expansions tell us that upward propagating solutions of the Helmholtz equation are

$$u(x, z) = \sum_{p=-\infty}^{\infty} \hat{a}_p e^{i\tilde{p}x + i\gamma_p^u z}.$$

- With this we can define the Transparent Boundary Conditions in the following way: we rewrite the solution in the upper layer as

$$u(x, z) = \sum_{p=-\infty}^{\infty} (\hat{a}_p e^{i\gamma_p^u a}) e^{i\tilde{p}x + i\gamma_p^u (z-a)} = \sum_{p=-\infty}^{\infty} \hat{\xi}_p e^{i\tilde{p}x + i\gamma_p^u (z-a)}.$$

Transparent Boundary Conditions

- We then observe that

$$\partial_z u(x, a) = \sum_{p=-\infty}^{\infty} (i\gamma_p^u) \hat{\xi}_p e^{i\tilde{p}x} =: T^u[\xi(x)],$$

which defines the order–one Fourier multiplier T^u .

- A similar procedure in the lower layer shows that we can write

$$\partial_z w(x, -b) = \sum_{p=-\infty}^{\infty} (-i\gamma_p^w) \hat{\psi}_p e^{i\tilde{p}x} =: T^w[\psi(x)],$$

for the order–one Fourier multiplier T^w .

Upward and Downward Propagating Solutions

- From these we state that upward-propagating solutions of the upper layer satisfy the Transparent Boundary Condition at $z = a$

$$\partial_z u(x, a) - T^u[u(x, a)] = 0, \quad z = a.$$

- Similarly, downward-propagating solutions in the lower layer satisfy the Transparent Boundary Condition at $z = -b$

$$\partial_z w(x, -b) - T^w[w(x, -b)] = 0, \quad z = -b.$$

Full Governing Equations

With these we now state the full set of governing equations as

$$\Delta u + 2i\alpha\partial_x u + (\gamma^u)^2 u = 0, \quad z > g(x),$$

$$\Delta w + 2i\alpha\partial_x w + (\gamma^w)^2 w = 0, \quad z < g(x),$$

$$u - w = \zeta, \quad z = g(x),$$

$$\partial_N u - i\alpha(\partial_x g)u - \tau^2 [\partial_N w - i\alpha(\partial_x g)w] = \psi, \quad z = g(x),$$

$$\partial_z u(x, a) - T^u[u(x, a)] = 0, \quad z = a,$$

$$\partial_z w(x, -b) - T^w[w(x, -b)] = 0, \quad z = -b,$$

$$u(x + d, z) = u(x, z),$$

$$w(x + d, z) = w(x, z).$$

Domain Decomposition Method

- We now write our governing equations in terms of surface quantities. For this we define the Dirichlet traces and their (outward) Neumann counterparts

$$\begin{aligned} U(x) &:= u(x, g(x)), & \tilde{U}(x) &:= -\partial_N u(x, g(x)), \\ W(x) &:= w(x, g(x)), & \tilde{W}(x) &:= \partial_N w(x, g(x)), \end{aligned}$$

- In terms of these our full governing equations are equivalent to the pair of boundary conditions,

$$\begin{aligned} U - W &= \zeta, \\ -\tilde{U} - (i\alpha)(\partial_x g)U - \tau^2 \left[\tilde{W} - (i\alpha)(\partial_x g)W \right] &= \psi. \end{aligned}$$

- The set of two equations and four unknowns can be closed by noting that the pairs $\{U, \tilde{U}\}$ and $\{W, \tilde{W}\}$ are connected, e.g., by DNOs

$$G : U \rightarrow \tilde{U}, \quad J : W \rightarrow \tilde{W}.$$

Interfacial Reformulation

The interfacial reformulation of our governing equations becomes

$$\mathbf{A}\mathbf{V} = \mathbf{R},$$

where

$$\mathbf{A} = \begin{pmatrix} I & -I \\ G + (\partial_x g)(i\alpha) & \tau^2 J - \tau^2 (\partial_x g)(i\alpha) \end{pmatrix},$$

$$\mathbf{V} = \begin{pmatrix} U \\ W \end{pmatrix}, \quad \mathbf{R} = \begin{pmatrix} \zeta \\ -\psi \end{pmatrix}.$$

Numerical Methods

- A variety of numerical algorithms have been devised for the simulation of these problems including Finite Difference, Finite Element, and Spectral Element methods.
- These methods suffer from the requirement that they discretize the **full volume** of the problem domain.
- We advocate the use of **surface methods**, especially the High-Order Perturbation of Surfaces (HOPS) methods:
 - provide the solution at the interface.
 - only discretize the layer interfaces.
 - are highly accurate, rapid, and robust.
- The HOPS methods are based on the foundational contributions of
 - Field Expansion (FE) method: Bruno & Reitich (1993).
 - Transformed Field Expansion (TFE) method: Nicholls & Reitich (1999).

Boundary and Frequency Perturbations

- We take a perturbative approach which makes two smallness assumptions:
 - 1 Boundary Perturbation: $g(x) = \varepsilon f(x)$, $\varepsilon \in \mathbf{R}$, $\varepsilon \ll 1$,
 - 2 Frequency Perturbation: $\omega = (1 + \delta)\underline{\omega}$, $\delta \in \mathbf{R}$, $\delta \ll 1$.
- The second of these assumptions has the following important consequences

$$k^q = (1 + \delta)\underline{k}^q, \quad \alpha = (1 + \delta)\underline{\alpha}, \quad \gamma^q = (1 + \delta)\underline{\gamma}^q,$$

for $q \in \{u, w\}$.

Transformed Field Expansions Method

- The method of Transformed Field Expansions (TFE) proceeds a domain-flattening change of variables prior to perturbation expansion.
- Focusing on the upper layer, the change of variable is

$$x' = x, \quad z' = a \left(\frac{z - g(x)}{a - g(x)} \right),$$

which maps the perturbed domain $\{g(x) < z < a\}$ to the separable domain $\{0 < z' < a\}$.

- A similar transformation occurs in the lower layer where the perturbed domain $\{-b < z < g(x)\}$ becomes $\{-b < z' < 0\}$.

Perturbation Expansions

- Provided f is sufficiently smooth, we will later show we will show the joint analytic dependence of $\mathbf{A} = \mathbf{A}(\varepsilon, \delta)$ and $\mathbf{R} = \mathbf{R}(\varepsilon, \delta)$ upon ε and δ , will induce a jointly analytic solution, $\mathbf{V} = \mathbf{V}(\varepsilon, \delta)$.
- In this case we may expand

$$\{\mathbf{A}, \mathbf{V}, \mathbf{R}\}(\varepsilon, \delta) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \{\mathbf{A}_{n,m}, \mathbf{V}_{n,m}, \mathbf{R}_{n,m}\} \varepsilon^n \delta^m,$$

and a calculation reveals that at every perturbation order (n, m) , we can find the $\mathbf{V}_{n,m}$ by solving

$$\begin{aligned} \mathbf{A}_{0,0} \mathbf{V}_{n,m} &= \mathbf{R}_{n,m} - \sum_{\ell=0}^{n-1} \mathbf{A}_{n-\ell,0} \mathbf{V}_{\ell,m} - \sum_{r=0}^{m-1} \mathbf{A}_{0,m-r} \mathbf{V}_{n,r} \\ &\quad - \sum_{\ell=0}^{n-1} \sum_{r=0}^{m-1} \mathbf{A}_{n-\ell,m-r} \mathbf{V}_{\ell,r}. \end{aligned}$$

Order (n, m)

- A brief inspection of the formulas for \mathbf{A} and \mathbf{R} , reveals that

$$\mathbf{A}_{0,0} = \begin{pmatrix} I & -I \\ G_{0,0} & \tau^2 J_{0,0} \end{pmatrix},$$

$$\mathbf{A}_{n,m} = \begin{pmatrix} 0 & 0 \\ G_{n,m} & \tau^2 J_{n,m} \end{pmatrix}$$

$$+ \delta_{n,1} \{1 + \delta_{m,1}\} (\partial_x f)(i\alpha) \begin{pmatrix} 0 & 0 \\ 1 & -\tau^2 \end{pmatrix}, \quad n \neq 0 \text{ or } m \neq 0,$$

$$\mathbf{R}_{n,m} = \begin{pmatrix} \zeta_{n,m} \\ -\psi_{n,m} \end{pmatrix}.$$

- $\delta_{n,m}$ is the Kronecker delta function and the forms for $\zeta_{n,m}$ and $\psi_{n,m}$ are known.

Numerical Approximation

- In our approximation we begin by truncating the Taylor series

$$\begin{aligned} \{\mathbf{A}, \mathbf{V}, \mathbf{R}\}(\varepsilon, \delta) &\approx \{\mathbf{A}^{N,M}, \mathbf{V}^{N,M}, \mathbf{R}^{N,M}\}(\varepsilon, \delta) \\ &:= \sum_{n=0}^N \sum_{m=0}^M \{\mathbf{A}_{n,m}, \mathbf{V}_{n,m}, \mathbf{R}_{n,m}\} \varepsilon^n \delta^m, \end{aligned}$$

where we must specify (i.) how the forms $\mathbf{A}_{n,m}$ are simulated, and (ii.) how the operator $\mathbf{A}_{0,0}$ is to be inverted.

- Regarding the forms $\mathbf{A}_{n,m}$, these boil down to the (n, m) -th corrections of the DNOs G and J , respectively, in a Taylor series expansion of each jointly in ε and δ . We will simulate these numerically.
- The inversion of $\mathbf{A}_{0,0}$ will follow from the proof of existence and uniqueness.

A Fourier/Chebyshev Collocation Discretization

- To show how we simulate $\mathbf{A}_{n,m}$, we will focus on the upper layer DNO, G . We begin by approximating

$$u(x, z; \varepsilon, \delta) \approx u^{N,M}(x, z; \varepsilon, \delta) := \sum_{n=0}^N \sum_{m=0}^M u_{n,m}(x, z) \varepsilon^n \delta^m.$$

- Each of these $u_{n,m}(x, z)$ are then simulated by a Fourier–Chebyshev approach which posits the form

$$u_{n,m}(x, z) \approx u_{n,m}^{N_x, N_z}(x, z) := \sum_{p=-N_x/2}^{N_x/2-1} \sum_{\ell=0}^{N_z} \hat{u}_{n,m,p,\ell} e^{i\tilde{p}x} T_\ell \left(\frac{2z-a}{a} \right),$$

where T_ℓ is the ℓ -th Chebyshev polynomial. The unknowns $\hat{u}_{n,m,p,\ell}$ are recovered by the collocation approach.

Equispaced Grid Points / Collocation Points

- As mentioned previously, the Fourier–Chebyshev approach posits the form

$$u_{n,m}(x, z) \approx u_{n,m}^{N_x, N_z}(x, z) := \sum_{p=-N_x/2}^{N_x/2-1} \sum_{\ell=0}^{N_z} \hat{u}_{n,m,p,\ell} e^{i\tilde{p}x} T_\ell \left(\frac{2z-a}{a} \right).$$

- More specifically, our HOPS/AWE algorithm requires $N_x \times N_z$ unknowns at every perturbation order, (n, m) .
- As our problem is d -periodic in the lateral direction, we will expand using a Fourier spectral method where we require N_x equally-spaced gridpoints.
- However, our problem is not z -periodic, so our strategy is to use a Chebyshev spectral method in the vertical direction. For this, we select $N_z + 1$ collocation points.

Simulation of DNOs

- With this we can simulate the upper layer DNO through

$$G(x; \varepsilon, \delta) \approx G^{N,M}(x; \varepsilon, \delta) := \sum_{n=0}^N \sum_{m=0}^M G_{n,m}(x) \varepsilon^n \delta^m.$$

- Here

$$G_{n,m}(x) \approx G_{n,m}^{N_x}(x) := \sum_{p=-N_x/2}^{N_x/2-1} \hat{G}_{n,m,p} e^{i\tilde{p}x},$$

and the $\hat{G}_{n,m,p}$ are recovered from the $\hat{u}_{n,m,p,\ell}$.

- We apply the same procedure to the lower layer DNO, J .

The Rayleigh Expansions

- Previously, we observed that solutions to the Helmholtz problem in the upper layer can be expressed in terms of Rayleigh expansions

$$u(x, z) = \sum_{p=-\infty}^{\infty} \hat{a}_p e^{i\tilde{p}x + i\gamma_p^u z}.$$

- For $p \in \mathbf{Z}$ we define

$$\tilde{p} := \frac{2\pi p}{d}, \quad \alpha_p := \alpha + \tilde{p}, \quad \gamma_p^u := \begin{cases} \sqrt{(k^u)^2 - \alpha_p^2}, & p \in \mathcal{U}^u, \\ i\sqrt{\alpha_p^2 - (k^u)^2}, & p \notin \mathcal{U}^u. \end{cases}$$

Propagating Modes

- We have

$$\gamma_p^u := \begin{cases} \sqrt{(k^u)^2 - \alpha_p^2}, & p \in \mathcal{U}^u, \\ i\sqrt{\alpha_p^2 - (k^u)^2}, & p \notin \mathcal{U}^u, \end{cases} \quad \mathcal{U}^u := \{p \in \mathbf{Z} \mid \alpha_p^2 < (k^u)^2\}.$$

- Components of $u(x, z)$ corresponding to $p \in \mathcal{U}^u$ propagate away from the layer interface, while those not in this set decay exponentially from $z = g(x)$.
- The latter are called evanescent waves while the former are propagating (defining the set of propagating modes \mathcal{U}^u) and carry energy away from the grating.

The Reflectivity Map

- With this in mind one defines the efficiencies

$$e_p^u := (\gamma_p^u / \gamma^u) |\hat{a}_p|^2, \quad p \in \mathcal{U}^u,$$

- and the Reflectivity Map as the sum of efficiencies in the upper layer

$$R := \sum_{p \in \mathcal{U}^u} e_p^u.$$

- Similar quantities can be defined in the lower layer, and with these the principle of conservation of energy can be stated for structures composed entirely of dielectrics

$$\sum_{p \in \mathcal{U}^u} e_p^u + \tau^2 \sum_{p \in \mathcal{U}^w} e_p^w = 1.$$

Energy Defect

- In this situation a useful diagnostic of convergence for a numerical scheme is the “Energy Defect”

$$D := 1 - \sum_{p \in \mathcal{U}^u} e_p^u - \tau^2 \sum_{p \in \mathcal{U}^w} e_p^w,$$

which should be zero for a purely dielectric structure.

Rayleigh Singularities (Wood's Anomalies)

- The Taylor series expansion for γ_p^q , $q \in \{u, w\}$, is

$$\gamma_p^q = \gamma_p^q(\delta) = \sum_{m=0}^{\infty} \gamma_{p,m}^q \delta^m.$$

- Recalling $\underline{\gamma}_p^q = (1 + \delta)\underline{\gamma}_p^q$, $\underline{k}^q = (1 + \delta)\underline{k}^q$ one finds

$$\underline{\alpha}_p^2 + (\underline{\gamma}_p^q)^2 = (\underline{k}^q)^2.$$

- When $\underline{\gamma}_p^q = 0$, the Taylor series expansion of $\gamma_p^q(\delta)$ is invalid. A Rayleigh singularity (or Wood's anomaly) occurs when $\underline{\alpha}_p^2 = (\underline{k}^q)^2$.
- Therefore, the permissible values of δ are constrained by this.

The Domain of Analyticity

- To guide our computations we explore this restriction on δ .
- In the upper layer, Rayleigh singularities occur when $\underline{\alpha}_p^2 = (\underline{k}^u)^2$ which implies

$$\underline{\omega} = \pm \frac{c_0}{n^u} \left\{ \underline{\alpha} + \frac{2\pi p}{d} \right\}, \quad \text{for any } p \in \mathbf{Z}.$$

- In the interest of maximizing our choice of δ we select a “mid–point” value of $\underline{\omega}$ which is as far away as possible from consecutive Rayleigh singularities

$$\underline{\omega}_q := \frac{c_0}{n^u} \left\{ \underline{\alpha} + \frac{2\pi(q + 1/2)}{d} \right\}.$$

- Our algorithm will expand in δ at the “mid–points” away from Rayleigh singularities.

Simulation: Reflectivity Map for Vacuum over Dielectric

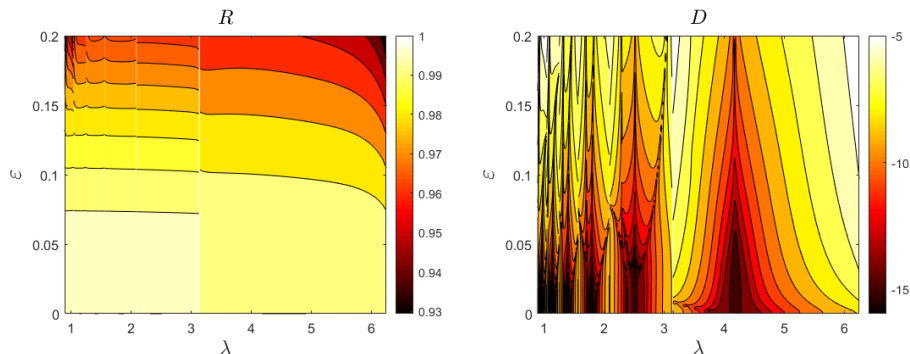


Figure 1: The Reflectivity Map, $R(\varepsilon, \delta)$, and energy defect D computed with our HOPS/AWE algorithm with Taylor summation. We set $N = M = 16$ and the parameter choices were $\alpha = 0$, $n^u = 1$, and $n^w = 1.1$.

Simulation: Reflectivity Map for Vacuum over Silver and Gold

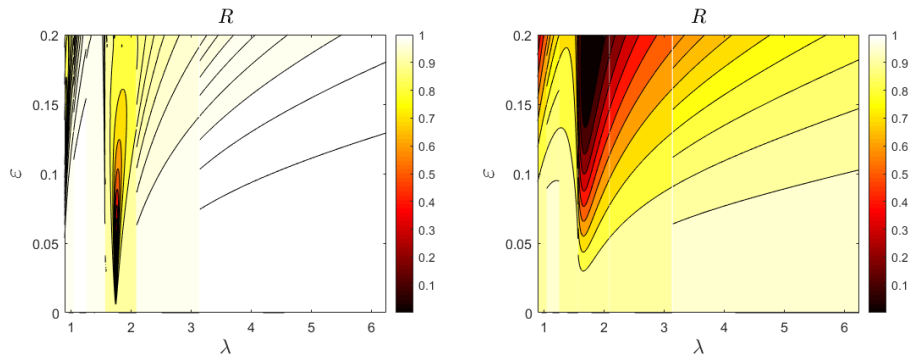


Figure 2: The Reflectivity Map, $R(\varepsilon, \delta)$, for silver (left) and gold (right) with Padé summation. We set $N = M = 15$ and parameter choices were $\alpha = 0$, $n^u = 1$, $n^w = 0.05 + 2.275i$ (left) and $n^w = 1.48 + 1.883i$ (right).

- The interfacial reformulation of our governing equations is $\mathbf{AV} = \mathbf{R}$ and the formulas for \mathbf{A} and \mathbf{R} at order (n, m) are

$$\mathbf{A}_{0,0} = \begin{pmatrix} I & -I \\ G_{0,0} & \tau^2 J_{0,0} \end{pmatrix},$$

$$\mathbf{A}_{n,m} = \begin{pmatrix} 0 & 0 \\ G_{n,m} & \tau^2 J_{n,m} \end{pmatrix}$$

$$+ \delta_{n,1} \{1 + \delta_{m,1}\} (\partial_x f)(i\alpha) \begin{pmatrix} 0 & 0 \\ 1 & -\tau^2 \end{pmatrix}, \quad n \neq 0 \text{ or } m \neq 0,$$

$$\mathbf{R}_{n,m} = \begin{pmatrix} \zeta_{n,m} \\ -\psi_{n,m} \end{pmatrix}.$$

- We will now establish the existence, uniqueness, and analyticity of solutions to $\mathbf{AV} = \mathbf{R}$.
- To accomplish this we will show the joint analytic dependence of $\mathbf{A} = \mathbf{A}(\varepsilon, \delta)$ and $\mathbf{R} = \mathbf{R}(\varepsilon, \delta)$ upon ε and δ , will induce a jointly analytic solution, $\mathbf{V} = \mathbf{V}(\varepsilon, \delta)$.

Theorem: Analyticity of Solutions [Kehoe, Nicholls 22]

Theorem

Given two Banach spaces X and Y , suppose that

H1 $\mathbf{R}_{n,m} \in Y$ for all $n, m \geq 0$, and there exists constants $B_R > 0, C_{R,N} > 0, C_{R,M} > 0, D_R > 0$ such that

$$\|\mathbf{R}_{n,m}\|_Y \leq C_{R,N} C_{R,M} B_R^n D_R^m,$$

H2 $\mathbf{A}_{n,m} : X \rightarrow Y$ for all $n, m \geq 0$, and there exists constants $B_A > 0, C_{A,N} > 0, C_{A,M} > 0, D_A > 0$ such that

$$\|\mathbf{A}_{n,m}\|_{X \rightarrow Y} \leq C_{A,N} C_{A,M} B_A^n D_A^m,$$

H3 $\mathbf{A}_{0,0}^{-1} : Y \rightarrow X$ for all $n, m \geq 0$, and there exists a constant $C_e > 0$ such that

$$\|\mathbf{A}_{0,0}^{-1}\|_{Y \rightarrow X} \leq C_e.$$

Theorem: Analyticity of Solutions (Continued)

Theorem (continued)

Then, given an integer $s \geq 0$, if $f \in C^{s+2}([0, d])$ then the linear system $\mathbf{A}\mathbf{V} = \mathbf{R}$ has a unique solution, $\sum_{n,m} \mathbf{V}_{n,m} \varepsilon^n \delta^m$, and there exist constants $B, C, D > 0$ such that

$$\|\mathbf{V}_{n,m}\|_{\mathcal{X}^s} \leq CB^n D^m,$$

for all $n, m \geq 0$. This implies that for any $0 \leq \rho, \sigma < 1$, $\sum_{n,m} \mathbf{V}_{n,m} \varepsilon^n \delta^m$ converges for all ε such that $B\varepsilon < \rho$, i.e., $\varepsilon < \rho/B$ and all δ such that $D\delta < \sigma$, i.e., $\delta < \sigma/D$.

Sketch of Proof

- First, we define the vector-valued spaces for $s \geq 0$

$$X^s := \left\{ \mathbf{V} = \begin{pmatrix} U \\ W \end{pmatrix} \middle| U, W \in H^{s+3/2}([0, d]) \right\},$$

$$Y^s := \left\{ \mathbf{R} = \begin{pmatrix} \zeta \\ -\psi \end{pmatrix} \middle| \zeta \in H^{s+3/2}([0, d]), \psi \in H^{s+1/2}([0, d]) \right\}.$$

- These have the norms

$$\|\mathbf{V}\|_{X^s}^2 = \left\| \begin{pmatrix} U \\ W \end{pmatrix} \right\|_{X^s}^2 := \|U\|_{H^{s+3/2}}^2 + \|W\|_{H^{s+3/2}}^2,$$

$$\|\mathbf{R}\|_{Y^s}^2 = \left\| \begin{pmatrix} \zeta \\ -\psi \end{pmatrix} \right\|_{Y^s}^2 := \|\zeta\|_{H^{s+3/2}}^2 + \|\psi\|_{H^{s+1/2}}^2.$$

Sketch of Proof (Continued)

- **Hypothesis H1:** Consider the Banach spaces $X = X^s$ and $Y = Y^s$. Our first task is to show that

$$\mathbf{R}_{n,m} = \begin{pmatrix} \zeta_{n,m} \\ -\psi_{n,m} \end{pmatrix},$$

is bounded in Y^s for any $s \geq 0$.

- Upon performing the boundary/frequency perturbations, we define

$$\mathcal{E}(x; \varepsilon, \delta) := e^{-i(1+\delta)\underline{\gamma}^u \varepsilon f(x)},$$

so that

$$\zeta(x) = \zeta(x; \varepsilon, \delta) = -\mathcal{E}(x; \varepsilon, \delta),$$

$$\psi(x) = \psi(x; \varepsilon, \delta) = \{i(1+\delta)\underline{\gamma}^u + i(1+\delta)\underline{\alpha}(\varepsilon \partial_x f)\} \mathcal{E}(x; \varepsilon, \delta).$$

- A joint Taylor expansion followed by an induction argument shows that $\|\zeta_{n,m}\|_{H^{s+3/2}}$ and $\|\psi_{n,m}\|_{H^{s+1/2}}$ are bounded. Therefore, $\|\mathbf{R}_{n,m}\|_{Y^s}$ is bounded.

Sketch of Proof (Continued)

- **Hypothesis H2:** Our next task is to show that the operators $G_{n,m}$ and $J_{n,m}$ in

$$\mathbf{A}'_{n,m} = \begin{pmatrix} 0 & 0 \\ G_{n,m} & \tau^2 J_{n,m} \end{pmatrix},$$

for the Taylor series expansions of the DNOs satisfy the appropriate bounds.

- For brevity, we will outline our technique for the Taylor expansion of the upper layer DNO, $G_{n,m}$.
- **Lemma (Algebra Property):** Given an integer $s \geq 0$, there exists a constant $\mathcal{M} = \mathcal{M}(s)$ such that if $f \in C^s([0, d])$ and $u \in H^s([0, d] \times [0, a])$ then

$$\|fu\|_{H^s} \leq \mathcal{M} \|f\|_{C^s} \|u\|_{H^s}.$$

Sketch of Proof (Continued)

- The bound on $G_{n,m}$ follows from
 - Applying the boundary and frequency perturbations followed by the TFE method results in the upper layer DNO problem

$$\begin{aligned} \Delta u_{n,m} + 2i\alpha \partial_x u_{n,m} + (\underline{\gamma}^u)^2 u_{n,m} &= F_{n,m}(x, z), & 0 < z < a, \\ u_{n,m}(x, 0) &= U_{n,m}(x), & z = 0, \\ \partial_z u_{n,m}(x, a) - T^u[u_{n,m}(x, a)] &= P_{n,m}(x), & z = a, \end{aligned}$$

where

$$G_{n,m}(f) = -\partial_z u_{n,m}(x, 0) + H_{n,m}(x).$$

- The **Algebra Property** establishes bounds on the non-homogeneous terms $F_{n,m}$, $P_{n,m}$, and $H_{n,m}$.
- With these, the **Elliptic Estimate** and an induction argument establishes

$$\|u_{n,m}\|_{H^{s+2}} \leq KB^n D^m,$$

for constants $K, B, D > 0$. This shows that the transformed upper field is jointly analytic with respect a boundary/frequency perturbation.

Sketch of Proof (Continued)

- The bound on $G_{n,m}$ follows from (continued)
 - ④ The bound on the upper layer DNO

$$G_{n,m}(f) = -\partial_z u_{n,m}(x, 0) + H_{n,m}(x),$$

then follows from the joint analyticity of the transformed upper field, $u_{n,m}$, an induction argument, and the fact that $H_{n,m}$ is bounded.

- ⑤ One finds

$$\|G_{n,m}\|_{H^{s+1/2}} \leq \tilde{K} \tilde{B}^n \tilde{D}^m,$$

for constants $\tilde{K}, \tilde{B}, \tilde{D} > 0$ which shows that $G_{n,m}$ is bounded. A similar argument works for the lower layer DNO, $J_{n,m}$, so that $\mathbf{A}_{n,m}$ is bounded and **H2** is satisfied.

Sketch of Proof (Continued)

- **Hypothesis H3:** Our final task is show that $\mathbf{A}_{0,0}^{-1}$ exists and the estimates and mapping properties of $\mathbf{A}_{0,0}^{-1}$ hold where $\mathbf{A}_{0,0}$ is defined by

$$\mathbf{A}_{0,0} = \begin{pmatrix} I & -I \\ G_{0,0} & \tau^2 J_{0,0} \end{pmatrix}.$$

- We define the operator

$$\Delta := G_{0,0} + \tau^2 J_{0,0} = (-i\gamma_D^u) + \tau^2(-i\gamma_D^w),$$

so that Δ^{-1} exists and that

$$\Delta : H^{s+3/2} \rightarrow H^{s+1/2}, \quad \Delta^{-1} : H^{s+1/2} \rightarrow H^{s+3/2}.$$

Sketch of Proof (Continued)

- Next, we write generic elements of X^s and Y^s as

$$\mathbf{V} = \begin{pmatrix} U \\ W \end{pmatrix} \in X^s, \quad \mathbf{R} = \begin{pmatrix} \zeta \\ -\psi \end{pmatrix} \in Y^s.$$

- Using the definitions of the norms of X^s and Y^s we find

$$\|\mathbf{A}_{0,0}\mathbf{V}\|_{Y^s}^2 \leq C \|\mathbf{V}\|_{X^s}^2,$$

so that $\mathbf{A}_{0,0}$ maps X^s to Y^s . Furthermore,

$$\left\| \mathbf{A}_{0,0}^{-1}\mathbf{R} \right\|_{X^s}^2 \leq C_{\Delta^{-1}} \|\mathbf{R}\|_{Y^s}^2,$$

which shows that $\mathbf{A}_{0,0}^{-1}$ maps Y^s to X^s .

- Thus, $\|\mathbf{A}_{0,0}^{-1}\|_{Y^s \rightarrow X^s}$ is bounded and the mapping properties hold.

Conclusion

We seek outgoing/bounded, periodic solutions of the scattering problem

$$\Delta u + 2i\alpha\partial_x u + (\gamma^u)^2 u = 0, \quad z > g(x),$$

$$\Delta w + 2i\alpha\partial_x w + (\gamma^w)^2 w = 0, \quad z < g(x),$$

$$u - w = \zeta, \quad z = g(x),$$

$$\partial_N u - i\alpha(\partial_x g)u - \tau^2 [\partial_N w - i\alpha(\partial_x g)w] = \psi, \quad z = g(x).$$

1 Numerical Algorithm

- DNOs, boundary/frequency perturbations, and COV through TFE
- Joint Taylor expansion followed by Fourier/Chebyshev collocation
- Simulated scattered energy through Reflectivity map

2 Joint Analyticity of Solutions

- Reformulate governing equations in terms of a linear system
- Sobolev space theory: Algebra Property and Elliptic Estimate

Future Work

- 1 Extend HOPS/AWE algorithm to multilayered surfaces with different material layers. Introduce a new DNO to handle the intermediate layers.
- 2 Implement parallel programming techniques to handle the computation of the intermediate layers.
- 3 Introduce multiple small perturbation parameters outside of an interfacial perturbation and a frequency perturbation. Extend the proof of analyticity to handle any finite number of perturbation parameters.
- 4 Develop techniques to expand around Rayleigh singularities where the Taylor series expansion is invalid.

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